University of Idaho
High School Mathematics Competition 2023

Division II Solutions

Division II, Problem 1

Alice can paint a wall by herself in 80 minutes. Bob needs 3 hours to paint the same wall by himself. If Alice and Bob work together, how long will it take for them to paint that wall? What fraction of the wall will Alice paint? Assume that they don't distract each other and each paints at the same rate they would if they were painting by themselves.

Solution: Alice paints $1/80$ of a wall per minute. Bob paints $1/180$ of a wall per minute. Together they paint $1/80 + 1/180 = 13/720$ of a wall per minute. Hence it takes $720/13 = 55.38$ minutes to paint the wall. Alice will paint $9/13 = 0.69$ of the wall.

Division II, Problem 2

Find a positive integer $n < 100$ so that $\frac{n^6}{2^n} \leq 10^{-6}$, together with an explanation by hand that your answer satisfies those conditions.

Solution: Equivalently, we may look for solutions to $(10n)^6 \leq 2^n$ or $10n \leq 2^{n/6}$. Let $n = 6k$ to only worry about integer powers of 2. So we may look for solutions to $60k \leq 2^k$. Since $60 < 64 = 2^6$, it is sufficient to find k such that $2^6 k \leq 2^k$, or $k \leq 2^{k-6}$. Finally, $2^4 = 16$, so $10 \leq 2^{10-6}$ and we may use $k = 10$, which translates to $n = 60$: $\frac{60^6}{2^{60}} \leq 10^{-6}$.

In fact, the least value of n which satisfies the inequality is $n = 55$, for which $\frac{n^6}{2^n} \approx 7.68 \times 10^{-7}$. There are many good solutions.

Division II, Problem 3

How many numbers x, with $0 \leq x \leq \pi/2$ are there so that

$$\frac{2n+2}{n^2+1} \sin x$$

is an integer for some integer n?

Solution: Since $0 \leq x \leq \pi/2$, it must be that $0 \leq \sin x \leq 1$. Also have $0 < \frac{2n+1}{n^2+1} < 2$. Therefore $0 \leq \frac{2n+2}{n^2+1} \sin x < 2$. The only integer values the expression can have are 0 or 1.

If $\frac{2n+2}{n^2+1} \sin x = 0$ then $\sin x = 0$ and $x = 0$.

If $\frac{2n+2}{n^2+1} \sin x = 1$, then $\sin x = \frac{n^2+1}{2n+2}$. This has one solution for each $n = 0, 1, 2$. When $n \geq 3$, the right hand side is greater than 1, so there is no solution.

Conclusion: there are 4 values of x.

Division II, Problem 4

Suppose we have a regular triangular pyramid (one where every face is an equilateral triangle of the same size) of volume $\frac{\sqrt{2}}{12}$. Construct a smaller regular triangular pyramid by using the centroids of the 4 faces as the vertices. What is the volume of the smaller pyramid?

Solution: Let $ABDC$ is the large pyramid, and let $abcd$ is the small pyramid, where a is the centroid of $\triangle BCD$, b is the centroid of $\triangle ACD$, and so on. Let E be the center of the edge CD. Consider $\triangle AEB$. Because b lies on AE with $|bE| = \frac{1}{3}|AE|$, and a lies on BE with $|aE| = \frac{|}{B}E|$, by the similarity of $\triangle AEB$ and $\triangle bEa$, we have $|ab| = \frac{1}{3}|AB|$. Hence the edge length of the smaller tetrahedron is 1/3 of the larger tetrahedron. Consequently, the smaller one have volume $\frac{\sqrt{2}}{(12)(27)} = \frac{\sqrt{2}}{324}$.

Division II, Problem 5

For the purposes of this problem, we say that a number n is **beautiful** if $2n$ is a perfect square and $3n$ is a perfect cube. Find all beautiful numbers less than 10000. (Hint: There are two of them, but you have to justify this fact.)

Solution: Let us write the prime factorization of n as $n = 2^\alpha 3^\beta m$, where m is a positive integer such that $gcd(m, 2) = gcd(m, 3) = 1$.

Since $2n$ is a square, $\alpha + 1$ and β are even, and $m = a^2$ for some a.

Since $3n$ is a cube, α and $\beta + 1$ are divisible by 3, and $m = b^3$ for some b.

Since $\alpha + 1$ is even and α is divisible by 3, $\alpha = 6k + 3$ for some k. Since β is even and $\beta + 1$ is divisible by 3, $\beta = 6\ell + 2$ for some ℓ. Furthermore, m must be a 6-th power.

This means $n = 2^3 3^2 2^{6k} 3^{6\ell} m$. Note $2^{6k} 3^{6\ell} m$ is a 6th power, so $n = 72m'$ for some 6th power m'. For $m' = 1^6 = 1$, we get $n = 72$. For $m' = 2^6 = 64$, we get $n = 4608$. For $m' = 3^6$ (or larger), we will have $n > 10000$.

Division II, Problem 6

Solve the equation

$$x^2 + \frac{1}{x^2} - 5 - \frac{5}{x} + 6 = 0.$$

Solution: Completing the square, we can rewrite the equation as $\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) + 4 = 0.$ Let $y = x + \frac{1}{x}$; then we have $y^2 - 5y + 4 = 0$. So, $y = 1, 4$. The equation $x + \frac{1}{x} = 1$ has no solution; The equation $x + \frac{1}{x} = 4$ has solutions $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$.

Division II, Problem 7

You are told that the polynomial $2x^4 + ax^2 + b$ is divisible by the polynomial $x^2 - 2x + 3$. Find a and b.

Solution: Write $2x^4 + ax^2 + b = 2(x^2 - 2x + 3)(x^2 + cx + d)$. Replacing x by $-x$ does not change the left hand side, so we have

$$(x^2 - 2x + 3)(x^2 + cx + d) = (x^2 + 2x + 3)(x^2 - cx + d)$$

for all x. Thus, $c = 2$ and $d = 3$ because the two factorizations must match. Therefore,

$$2x^4 + ax^2 + b = 2[(x^2 + 3)^2 - 4x^2] = 2x^4 + 4x^2 + 18.$$

Hence, $a = 4$ and $b = 18$.

Division II, Problem 8

Yana and Zoe have a jar with 2023 marbles, which they use to play a game. Each player, on her turn, gets to take at least 1 and at most 23 marbles out of the jar. The player who takes the last marble wins. (For example, if there are 17 marbles left in the jar, the player whose turn it is can win by taking all the marbles.) Suppose Yana goes first. Describe a strategy that Yana can use to make sure she wins the game no matter what Zoe does. In particular, how many marbles should Yana take on her first turn?

Solution: If Zoe starts their turn with 24 marbles in the jar, then Yana will win, since no matter how many marbles Zoe takes, Yana can take the rest. Similarly, if Zoe starts their turn with some multiple of 24 marbles in the jar, then, if Zoe takes x marbles, Yana can take $24 - x$ marbles so that there is still some multiple of 24 marbles in the jar. Hence, since $2023 = 84(24) + 7$, Yana should take 7 marbles to leave Zoe with a multiple of 24, and then follow the strategy outlined above.