University of Idaho
High School Mathematics Competition 2023

Division I Solutions

Division I, Problem 1

Alice can paint a wall by herself in one hour. Bob needs 90 minutes to paint the same wall by himself. If Alice and Bob work together, how long will it take for them to paint that wall? What fraction of the wall will Alice paint? Assume that they don't distract each other and each paints at the same rate they would if they were painting by themselves.

Solution: Alice paints 1 wall per hour. Bob paints 2/3 of a wall per hour. Together, they paint 5/3 of a wall per hour, so they paint one wall in 3/5 of an hour, or 36 minutes. Alice paints 3/5 of the wall and Bob paints 2/5.

Division I, Problem 2

Find all ways that the number 500 can be expressed as the sum of some number of consecutive positive integers. (For example, $39 = 19 + 20$ and $39 = 12 + 13 + 14$ are two ways of writing 39 as the sum of consecutive integers.)

Solution: If k is odd, then the sum of k consecutive integers will be ka, where a is an integer (and the middle integer of the sequence). If k is even, then the sum of k consecutive integers will be kb, where $b - 1/2$ is an integer (and b is the average of the two middle integers of the sequence). Hence we need to know about odd integers k where k divides 500 and about even integers where k does not divide 500 but $2k$ divides 500. The possible k are $5, 25, 125, 8, 40, 200, 1000$. Note the sequences with $k = 125, 40, 200, 1000$ will use some negative integers. Hence the answers are $98+99+100+101+102 = 500$, $13+14+\cdots+37 = 500$, and $59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 = 500$.

Division I, Problem 3

Consider the parabola (given by an equation $y = ax^2 + bx + c$ for some numbers a, b, and c which we are not telling you) that goes through the points $(2023, -5)$, $(2035, -5)$, and $(2029, 4)$. Where does this parabola cross the x-axis?

Solution: The parabola is $y = -((x - 2029)/2)^2 + 4$ This has roots at $x = 2025$ and $x = 2033$. (Note: you could probably also figure this out by drawing the graph.)

Division I, Problem 4

An isoceles triangle with side lengths 2, 4, and 4 is inscribed in a circle. Find the radius of the circle.

Solution: Label the vertices of the triangle A, B, C, with $|AB| = |AC| = 4$, and $|BC| = 2$. Construct the lines that bisect $\overline{AB}$ and $\overline{BC}$ at their midpoints D and E, respectively. The segment bisectors intersect at a point O. SAS applied to triangles ΔAOD, ΔBOD, and to ΔBOE, ΔCOE, shows O to be equidistant from points A, B, and C; let $r = |OA| = |OB| = |OC|$. Thus, O is the center of the circle, and the goal is to calculate r.

Now apply the Pythagorean Theorem to ΔAEB to find $|AE| = \sqrt{|AB|^2 - |BE|^2} = \sqrt{15}$. Finally, the triangle is isosceles so substitute $|OE| = |AE| - |AO| = \sqrt{15} - r$ into the Pythagorean Theorem for ΔOEB, yielding $r^2 = 1 + \left(\sqrt{15} - r\right)^2$, with the solution $r = 8/\sqrt{15}$.

Division I, Problem 5

A sequence of numbers is constructed as follows. The first number is 2023^{2023}. If a number n in the sequence is a perfect square, then the next number in the sequence is $\sqrt{n}$. Otherwise, the next number is $n+1$. What is the smallest number in this sequence (which may happen more than once)?

Solution: Starting from any number n, we will add to n until we reach a perfect square, at which point we will take a square root. This square root will be smaller than n as long as $n \geq 2$. Hence, no matter where we start, the sequence will keep getting smaller until it reaches 2, at which point it will repeat $2, 3, 4, 2, 3, 4, \ldots$.

Division I, Problem 6

Find two rectangles R and S, both with integer side lengths, so that the perimeter of R is twice the perimeter of S and the area of S is twice the area of R.

Solution: Suppose R is $x \times y$ and S is $a \times b$.

Suppose that

$$\begin{aligned} 2a + 2b &= 4x + 4y, \\ \text{and } 2ab &= xy. \end{aligned}$$

Let's try $a = 1$. Then, solving for b in equation 1, $2b = 4x + 4y - 1$. Substituting $2b$ into equation 2, $xy = 4x + 4y - 2$. Solving this for x, it equals $\frac{4y-2}{y-4}$. Say $y = 5$, to make sure x is an integer. Then, $x = 18$, and $b = 2\,(18) + 2\,(5) - 1 = 45$. Checking, the perimeter and area of rectangle 1×45 are 92 and 45, respectively, and the perimeter and area of rectangle 18×5 are 46 and 90, respectively.

There are many solutions can be easily checked.

Division I, Problem 7

What is the highest power of 2 that divides $101 \times 102 \times 103 \times \cdots \times 200$?

Solution: Of these numbers, 50 are divisible by 2, 25 are divisible by 4, 13 are divisible by 8, 6 are divisible by 16, 3 are divisible by 32, 2 are divisible by 64, and 1 is divisible by 128. This adds up to 100. (If you don't see why we add all those numbers, we can do this the long way and add $(50-25)+2(25-13)+3(13-6)+4(6-3)+5(3-2)+6(2-1)+7 = 100$.

Division I, Problem 8

Yana and Zoe have a jar with 2023 marbles, which they use to play a game. Each player, on her turn, gets to take at least 1 and at most 23 marbles out of the jar. The player who takes the last marble wins. (For example, if there are 17 marbles left in the jar, the player whose turn it is can win by taking all the coins.) Suppose Yana goes first. Describe a strategy that Yana can use to make sure she wins the game no matter what Zoe does. In particular, how many marbles should Yana take on her first turn?

Solution: If Zoe starts their turn with 24 marbles in the jar, then Yana will win, since no matter how many marbles Zoe takes, Yana can take the rest. Similarly, if Zoe starts their turn with some multiple of 24 marbles in the jar, then, if Zoe takes x marbles, Yana can take $24 - x$ marbles so that there is still some multiple of 24 marbles in the jar. Hence, since $2023 = 84(24) + 7$, Yana should take 7 marbles to leave Zoe with a multiple of 24, and then follow the strategy outlined above.