University of Idaho
High School Mathematics Competition 2022
Divisions I, II (Corrected)

Competition Guidelines

- Students may only use paper and a writing implement. Other aids (such as calculators, phones, laptops, and slide rules) are not allowed.
- The competition consists of 10 questions. Teams will have 2 hours.
- Each problem will be graded out of 10 points, for a maximum of 100 points.
- Every solution should contain a succinct explanation of why the answer is correct or how the team arrived at the answer. Even for a correct answer, this explanation will be taken into consideration in the grading.
- Write your team number in the space provided on each problem. (The problems will be split up for grading.) Please DO NOT write any other identifying information (such as your name or school) on any pages turned in.
- If you require additional pages to write your solution to a problem, use blank paper and staple it to the problem page.
- In the highly unlikely event of a tie score, the team that submitted the problems earlier will be deemed the winner (of the tie).

Division I Problems

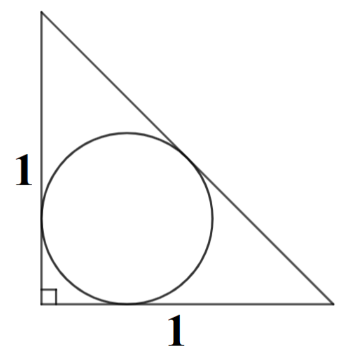
(1) Write

$$\frac{2001^3 - 1986^3 - 15^3}{(2001)(1986)(15)}$$

as an integer or as a fraction in lowest terms.

(2) Because of winds, an airplane travels from City A to City B at 600 miles per hour but makes the trip from City B to City A at 400 miles per hour. What would be the average speed over a round trip between City A and City B? Furthermore, explain with algebra why the answer does not depend on the distance between the two cities. (Note: The answer is NOT 500 miles per hour.)

(3) Consider an isosceles right triangle where each leg has length 1. What is the diameter of the inscribed circle?



(4) Put the numbers 2^{35}, 5^{15}, and 6^{14} in increasing order.

(5) How many diagonals are there in a regular 19-gon? (A diagonal is a line connecting two non-adjacent vertices.)

(6) Let

$$S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n.$$

What is

$$S_1 + S_2 + \cdots + S_{2021}?$$

(7) Consider the expression

$$\frac{(x^2-3)(x^2-4)\cdots(x^2-99)}{(x^2-4)(x^2-9)\cdots(x^2-81)}.$$

Find all integers x where this expression is negative.

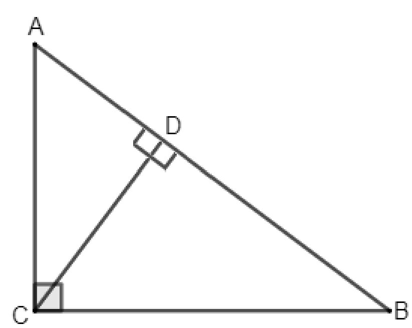
The intent of the problem is for the above expression to NOT be interpreted literally. The denominator is intended only to indicate the terms that should be canceled from the numerator, so the expression should be

$$(x^2-3)(x^2-5)\cdots(x^2-8)(x^2-10)\cdots.$$

The judges will NOT be impressed if you write that the expression is undefined for $x=2$ (although, literally, the original expression is undefined for $x=2$).

(8) Let $x=\sqrt{2}+\sqrt{5}$. Find some integers b, c, d, and e such that $x^4+bx^3+cx^2+dx+e=0$.

(9) The side lengths of a right triangle and its altitude are all integers. What are the areas of the smallest possible such right triangles?



(10) Alice and Bob are really bored and decide to play the following game. They have a long piece of wood with 2022 holes, all in a line. At the start of the game, there are 1000 pegs in the first 1000 holes to the left. Each player in turn moves one peg into an empty hole to its right. The player who makes the last move so that all 1000 pegs are in the last 1000 holes to the right wins. Alice goes first. Give a strategy Bob can use to win the game no matter what moves Alice makes.

Division II Problems

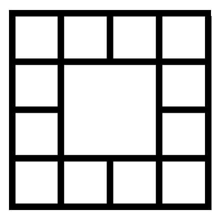
(1) Because of winds, an airplane travels from City A to City B at 600 miles per hour but makes the trip from City B to City A at 400 miles per hour. What would be the average speed over a round trip between City A and City B? Furthermore, explain with algebra why the answer does not depend on the distance between the two cities. (Note: The answer is NOT 500 miles per hour.)

(2) How many solutions does $\cos(2x) = \cos(x)$ have with $0 \leq x \leq \pi$?

(3) Simplify

$$\frac{3^{\frac{1}{\ln 3}} \cdot 4^{\frac{1}{\ln 4}}}{20^{\frac{1}{\ln 20}} \cdot 22^{\frac{1}{\ln 22}}}.$$

(4) Fill in the 12 cells along the edge of a 4×4 table with the numbers 1 through 12 (each used exactly once) so that the numbers along each edge has the same sum.



(5) Put the following numbers in increasing order: 3^{4^5}, 3^{5^4}, 4^{3^5}, 4^{5^3}.

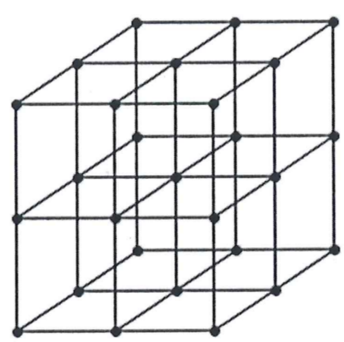
(6) Two circles are put inside a unit square without overlapping. Find the maximum sum of their radii.

(7) Let

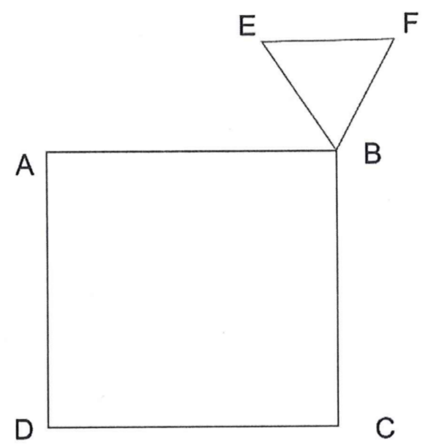
$$A = 1 + 3 + 3^2 + 3^3 + \cdots + 3^{2022}.$$

Find the remainder when A is divided by 40.

(8) An ant is climbing along the lattice of bars drawn below. It starts from the bottom left front corner and goes to the top right back corner, and always goes up, right, or back. (This means it always climbs on exactly 6 different bars.) How many different paths can this ant take? (Paths are considered the same if they use the same bars.)



(9) Let $ABCD$ be a square and BEF an equilateral triangle as in the picture below. Assume that $EF||AB$ and $AC = AF$. Compute $\tan(\angle BAF)$.



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