

University of Idaho
High School Mathematics Competition 2022

Division I Solutions

Division I, Problem 1

Write

$$\frac{(2001^3 - 1986^3 - 15^3)}{(2001)(1986)(15)}$$

as an integer or as a fraction in lowest terms.

Solution: Set $a = 1986$ and $b = 15$. Then the fraction becomes

$$\frac{(a+b)^3 - a^3 - b^3}{(a+b)ab} = \frac{a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3}{a^2b + ab^2} = \frac{3a^2b + 3ab^2}{a^2b + ab^2} = 3.$$

Division I, Problem 2

Because of winds, an airplane travels from City A to City B at 600 miles per hour but makes the trip from City B to City A at 400 miles per hour. What would be the average speed over a round trip between City A and City B? Furthermore, explain with algebra why the answer does not depend on the distance between the two cities. (Note: The answer is NOT 500 miles per hour.)

Solution: Let d be the distance between the two cities in miles. Then the trip from A to B takes $d/600$ hours and the trip from B to A takes $d/400$ hours. This makes the average speed

$$\frac{2d}{\frac{d}{600} + \frac{d}{400}} = \frac{2d}{\frac{2d}{1200} + \frac{3d}{1200}} = \frac{2400}{5} = 480.$$

Division I, Problem 3

Consider an isosceles right triangle where each leg has length 1. What is the diameter of the inscribed circle?

Solution: Label the vertices of the triangle A , B , and C , with A the right angle. Let O be the center of the inscribed circle, and D on AB , E on AC , and F on BC the tangent points where the inscribed circle touches the edges of the triangle. Notice that $ADOE$ forms a square of side length the radius of the inscribed circle. Let x be the radius of the circle; then AO has length $\sqrt{2}x$, and AF has length $\sqrt{2}x + x$. On the other hand, we know AF has length $\sqrt{2}/2$. Hence, we get $x = \sqrt{2}/(2(\sqrt{2} + 1))$, so the diameter is $\sqrt{2}/(\sqrt{2} + 1)$.

Division I, Problem 4

Put the numbers 2^{35} , 5^{15} , and 6^{14} in increasing order.

Solution: Since $32 < 36$, we have $2^{35} = (2^5)^7 < (6^2)^7 = 6^{14}$. Since $125 < 128$, $5^{15} = (5^3)^5 < (2^7)^5 = 2^{35}$. Therefore, we have $5^{15} < 2^{35} < 6^{14}$.

Division I, Problem 5

How many diagonals are there in a regular 19-gon? (A diagonal is a line connecting two non-adjacent vertices.)

Solution: An arbitrary n -gon has $\frac{n(n-3)}{2}$ diagonals, since each of the n vertices has a diagonal with $n-3$ other vertices, but this counts every diagonal twice, once at each end. Plugging in $n=19$ gives the answer of 152 diagonals.

Division I, Problem 6

Let

$$S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n.$$

What is

$$S_1 + S_2 + \cdots + S_{2021}?$$

Solution: Notice that $S_1 = 1$, $S_2 = -1$, $S_3 = 2$, $S_4 = -2$, and so on. This means $S_1 + S_2 + \cdots + S_{2021} = S_{2021} = (2021 + 1)/2 = 1011$.

Division I, Problem 7

Consider the expression

$$\frac{(x^2 - 3)(x^2 - 4) \cdots (x^2 - 99)}{(x^2 - 4)(x^2 - 9)(x^2 - 16) \cdots (x^2 - 81)}.$$

Find all the integers x where this expression negative?

Solution: When $|x| \geq 10$, all the factors are positive, so the expression is positive. For $|x| = 9$, the factors $(x^2 - a)$ for $a = 82, \dots, 99$ are negative, and there are an even number of such factors. Similarly, for $|x| = 8$, the above factors, as well as the factors $(x^2 - b)$ for $b = 65, \dots, 80$ are negative, and there are again an even number of such factors. The same argument applies until we get to $|x| = 1$, where the only additional factor is $(x^2 - 3)$. Hence, the only integers where this expression is negative are $x = -1, 0, 1$.

Division I, Problem 8

Let $x = \sqrt{2} + \sqrt{5}$. Find some integers b, c, d , and e such that $x^4 + bx^3 + cx^2 + dx + e = 0$.

Solution: We have

$$x - \sqrt{2} = \sqrt{5},$$

so

$$(x - \sqrt{2})^2 = 5,$$

so

$$x^2 - 2\sqrt{2}x + 2 = 5.$$

We can now rearrange this equation to

$$x^2 - 3 = 2\sqrt{2}x.$$

Squaring both sides again, we get

$$x^4 - 6x^2 + 9 = 8x^2,$$

so

$$x^4 - 14x^2 + 9 = 0.$$

Hence, $b = 0$, $c = -14$, $d = 0$, and $e = 9$ is a solution. (There are other solutions.)

Division I, Problem 9

The sides of a right triangle and its altitude are all integers. What is the area of the smallest possible such right triangle? What is the area of the smallest possible such right triangle not similar to the previous one?

Solution: Let a , b , and c be the (lengths of the) sides of the triangle, with c the hypotenuse. Let A be the angle opposite a and so on, and let D be the foot of the altitude, with d the length of the altitude CD . Note the triangle ABC is similar to the triangle CBD (and the triangle ACD), so $b/d = c/a$, which means $d = ab/c$. Hence for all the lengths to be integers, we must have a, b, c be a multiple of a Pythagorean triple, with c dividing ab . The smallest such solution is with $c = 5(5)$, so $a = 5(3)$ and $b = 5(4)$, giving an area of 150. Using the next Pythagorean triple $(5, 12, 13)$, we get $c = 13(13)$, $a = 13(5)$, and $b = 13(12)$, with area 5070.

Division I, Problem 10

Alice and Bob are really bored and decide to play the following game. They have a long piece of wood with 2022 holes, all in a line. At the start of the game, there are 1000 pegs in the first 1000 holes to the left. Each player in turn moves one peg into an empty hole to its right. The player who makes the last move so that all 1000 pegs are in the last 1000 holes to the right wins. Alice goes first. Give a strategy that Bob can use to win the game no matter what moves Alice makes.

Solution: Think of the board as being divided into pairs of holes (whether filled with a peg or not), with Holes 1 and 2 being partners, Holes 3 and 4 partners, and so on. For every move Alice makes, Bob moves the partner of the peg Alice moved to the partner of the hole Alice moved that peg to. For example, if Alice moves the peg in Hole 207 to Hole 1756, then Bob follows by moving the peg in Hole 208 to Hole 1755. Since Bob always has a move, Alice could not win this game, and so Bob will eventually win.